1999 SOLUTIONS, PART II

Problem 1. a) 167,166,167,166,167,166,167,166,167,166,167,168

b) If this were possible, there would be six tables with even numbers of cupcakes and six tables with odd numbers of cupcakes. The sum of six odd numbers and six even numbers is even. However, 1999 is odd.

Problem 2. Complete triangle ABC to a rectangle $ABCD$ and draw PD and QD. In the parallelogram $CPDQ$, the sum of the squares of the diagonals $CD^2+PQ^2=AB^2+AB^2/9$ equals the sum of the squares of the four sides $2(CP^2+CQ^2)$. Therefore $AB^2+AB^2/9=2\cdot 5$ and $AB=3$.

Problem 3. If not, then no two are equal. Without loss of generality assume that c is between a and b. Then $|P(a)-P(b)|=|c-b|<|b-a|$. It is easy to show that $b-a$ is a factor of $P(b)-P(a)$, a contradiction.

Problem 4. Since there are only 1999 colors, among every 2000 points there are two of the same color. There are $N=1999^{2000}$ ways of coloring a string of 2000 points. Consider the following $N+1$ strings of 2000 points $(k,1),(k,2),...,(k,2000)$, $k=0,1,...,N$. Since there are $N+1$ of them, the coloring pattern is the same for at least two of the strings. Each of the two strings has at least two points of the same color. The four points form a rectangle.

Problem 5. Brutus wins as follows:

(i) If Caesar leaves a single $(2n+1)$ by 1999 rectangle, Brutus leaves a single $(2n+1)$ by $(2n+1)$ square reducing the situation to a smaller odd square.

(ii) If Caesar leaves a single $2n$ by 1999 rectangle, Brutus leaves two n by 1999 rectangles and plays a mirror image strategy henceforth.

(iii) If Caesar leaves an m by 1999 rectangle and an n by 1999 rectangle with $m>n$, Brutus leaves two n by 1999 rectangles and plays a mirror image strategy henceforth.