42$^{\text{nd}}$ ANNUAL UNIVERSITY OF MARYLAND
HIGH SCHOOL MATHEMATICS COMPETITION
PART II
December 1, 2021, 1:00–3:00 pm

Instructions: Solutions to different problems should go on separate pages. Write down your name and the problem number on the upper left corner of all pages that you submit for grading. Show your work and justify all of your steps. Submit only what you want to be graded; no blank paper or scratch paper.

NO CALCULATORS
2 hours

1. The coins in Merryland all have different integer values: there is a single 1 cent coin, a single 2 cent coin, etc. What is the largest number of coins that a resident of Merryland can have if we know that their total value does not exceed 2021 cents?

2. For every positive integer k let
$$a_k = \left(\sqrt{\frac{k+1}{k}} + \frac{\sqrt{k+1}}{k} - \frac{1}{k} - \sqrt{\frac{1}{k}}\right)$$
Evaluate the product $a_4 a_5 \cdots a_{99}$. Your answer must be as simple as possible.

3. Prove that for every positive integer n there is a permutation $a_1, a_2, \ldots, a_n$ of $1, 2, \ldots, n$ for which $j + a_j$ is a power of 2 for every $j = 1, 2, \ldots, n$.

4. Each point of the 3-dimensional space is colored one of five different colors: blue, green, orange, red, or yellow, and all five colors are used at least once. Show that there exists a plane somewhere in space which contains four points, no two of which have the same color.

5. Suppose $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$ are real numbers. Let C_n be the union of n intervals as below:
$$C_n = [a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_n, b_n].$$
We say C_n is minimal if there is a subset W of real numbers $\mathbb{R}$ for which both of the following hold:
   (a) Every real number r can be written as $r = c + w$ for some c in C_n and some w in W; and
   (b) If D is a subset of C_n for which every real number r can be written as $r = d + w$ for some d in D and some w in W, then $D = C_n$.

   (i) Prove that every interval $C_1 = [a_1, b_1]$ is minimal.
   (ii) Prove that for every positive integer n, the set C_n is minimal.