THE 42^{nd} ANNUAL (2021) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART II SOLUTIONS

1. If they have n coins, we must have

$$1+2+\cdots+n<2021 \Rightarrow \frac{n(n+1)}{2}<2021 \Rightarrow n(n+1)<4042.$$

We notice that $63\times 64<4042$ and $64\times 65>4042$. Therefore, the answer is $n=63$.

2. Re-writing a_k we obtain

$$a_k=\frac{\sqrt{k(k+1)}+\sqrt{k+1}-1-\sqrt{k}}{k}=\frac{(\sqrt{k}+1)(\sqrt{k+1}-1)}{k}.$$

Multiplying by the conjugate of $\sqrt{k+1}-1$ we obtain the following:

$$a_k=\frac{(\sqrt{k}+1)(\sqrt{k+1}-1)(\sqrt{k+1}+1)}{k(\sqrt{k+1}+1)}=\frac{\sqrt{k}+1}{\sqrt{k+1}+1}.$$

Therefore,

$$a_4\cdots a_{99}=\prod_{k=4}^{99}\frac{\sqrt{k}+1}{\sqrt{k+1}+1}=\frac{(\sqrt{4}+1)\cdots(\sqrt{99}+1)}{(\sqrt{5}+1)\cdots(\sqrt{100}+1)}=\frac{3}{11}.$$

The answer is 3/11.

3. We will prove this by induction on n.

For $n=1$, the permutation $a_1=1$ gives $a_1+1=2$ which is a power of 2.

Let $n\geq 2$ be an integer, and assume the claim is true for all integers less than n. Suppose 2^k is the least power of 2 exceeding n, i.e. $2^{k-1}\leq n<2^k$. Since $2^k\leq 2n$ we have $2^k-n\leq n$. For simplicity let $m=2^k-n-1$. By inductive hypothesis, there is a permutation $a_1,\ldots,a_m$ of $1,\ldots,m$ for which a_j+j is a power of 2 for all $j=1,\ldots,m$.

Now, let $a_j=2^k-j$ for all $j=m+1,\ldots,n$. Note that $a_j+j=2^k$ is a power of 2. Furthermore, these values of a_j are $n-m$ consecutive integers starting with $2^k-m-1=n$ and decreasing until $2^k-n=m+1$. This means $a_1,\ldots,a_n$ is a permutation of $1,\ldots,n$ satisfying the given conditions.

4. Suppose on the contrary there is no plane that contains four points of different colors. We will obtain a contradiction.

Note that every line contains at most two colors, because if a line ℓ contains at least three colors, ℓ along with a point of a fourth color would lie on a plane which contains four points of different colors.

Now, assume B, G, O, R, Y are points with five different colors blue, green, orange, red, yellow, respectively. If line BG intersects the plane $\mathcal{P}$ containing O, R, Y, this intersection point must be blue or green, since it is on the line BG, but that means $\mathcal{P}$ contains four points of different colors. Therefore, BG must be parallel to $\mathcal{P}$. Let $\mathcal{Q}$ be the plane containing BG that is parallel to $\mathcal{P}$. If X is a blue point outside $\mathcal{Q}$, then XG intersects $\mathcal{P}$ which is a contradiction using a similar argument to above. Similarly there are no green points outside of $\mathcal{Q}$. Therefore, all green and blue points must lie on a single plane $\mathcal{Q}$. With a similar argument each pair of colors must be on a single plane, which means all points of the 3-dimensional space must lie on at most $\binom{5}{2} = 10$ planes, which is a contradiction.

5. For every two subsets X, Y of $\mathbb{R}$ and every real number r define

$$r + X = \{r + x \mid x \in X\}, \text{ and } X + Y = \{x + y \mid x \in X, \text{ and } y \in Y\}.$$

We will first prove the following claims:

Claim 1: For every real number r and every interval (a, b) we have $r + (a, b) = (r + a, r + b)$.

Proof. Every element of $r+(a, b)$ is of the form $r+x$, where $a < x < b$ and thus $r+a < r+x < r+b$. Which means $r + x \in (r + a, r + b)$. If y is an element of $(r + a, r + b)$, then $r + a < y < r + b$ which means $a < y - r < b$ and hence y can be written as $r + y - r$, where $y - r$ is in (a, b).

Claim 2. For every interval (a, b) of length more than $2(b_n - a_1)$, there is an interval (x, y) for which $C_n + (x, y) = (a, b)$.

Proof. Let $x = a - a_1$, and $y = b - b_n$. Every element of $C_n + (x, y)$ is strictly between $a_1 + x = a$ and $b_n + y = b$. On the other hand, $C_n + (x, y)$ contains $a_1 + (x, y) = (a_1 + x, a_1 + y) = (a, a_1 + b - b_n)$ as well as $b_n + (x, y) = (b_n + a - a_1, b)$. If we show $b_n + a - a_1 < a_1 + b - b_n$, then all numbers in the interval (a, b) will be covered by $C_n + (x, y)$. This inequality is equivalent to $2(b_n - a_1) < b - a$, which is true by assumption.

Now, we will create the set W as follows: Let r be a real number larger than $2(b_n - a_1)$. Consider the set $A = \{r, 2r, \ldots, nr\}$. Note that

$$C_n + A = [a_1 + r, b_1 + r] \cup [a_2 + r, b_2 + r] \cup \cdots \cup [a_n + nr, b_n + nr] \qquad (*)$$

We see that $b_n + kr < a_1 + (k + 1)r$, since $b_n - a_1 < r$. Therefore, all intervals in $(*)$ are disjoint, which means every element of $[a_k + kr, b_k + kr]$ appears exactly once as a sum of an element of C_n and an element of A. Furthermore, the length of the interval $(b_k + kr, a_{k+1} + (k+1)r)$ is $a_{k+1} - b_k + r > r$ which is larger than $2(b_n - a_1)$. By Claim 2, there is an interval I_k for which

$$C_n + I_k = (b_k + kr, a_{k+1} + (k + 1)r) \qquad (**)$$

Set

$$B = A \cup \bigcup_{k=1}^{n-1} I_k.$$

Combining $(*)$ with $(**)$ we conclude that $C_n + B = [a_1 + r, b_n + nr]$ and every element of $[a_k + kr, b_k + kr]$ has a unique representation as a sum of an element of C_n and an element of B. Take

$$W = B \cup (-\infty, a_1 + r - b_n) \cup (b_n + nr - a_1, \infty).$$

We know $C_n + B$ gives us $[a_1 + r, b_n + nr]$ and all elements of C_n are necessary to get to this interval. Since the smallest element of C_n is a_1 and its largest element is b_n,

$$C_n + (-\infty, a_1 + r - b_n) = (-\infty, a_1 + r), \text{ and } C_n + (b_n + nr - a_1, \infty) = (b_n + nr, \infty).$$

This shows $C_n + W = \mathbb{R}$ and all elements of C_n are necessary in order to obtain all real numbers.