THE 43rd ANNUAL (2022) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART I   MULTIPLE CHOICE

For each of the following 25 questions, carefully blacken the appropriate box on the answer sheet with a #2 pencil. Do not fold, bend or write stray marks on either side of the answer sheet. Each correct answer is worth 6 points. Two points are awarded for answers left blank. Zero points are given for incorrect answers. Note that wild guessing is apt to lower your score. When the exam is over, give your answer sheet to the proctor. You may keep your copy of the questions.

NO CALCULATORS

75 MINUTES

1. One hundred people are stranded on a desert island and they have exactly enough cookies that each person can have one cookie per day for 22 days. But before they can get started, N more people join them. Now there are exactly enough cookies for each person to have one cookie per day for 20 days. What is N?

   a. 10   b. 8   c. 6   d. 4   e. 2

2. Little Red Riding Hood draws the line $y = x + 6$ using a red pen. Little Boy Blue draws the parabola $y = x^2$ using a blue pen. How many purple points are produced? (A purple point is, of course, a point where the line intersects the parabola.)

   a. 0   b. 1   c. 2   d. 3   e. infinitely many

3. A rabbit and a terrapin are racing from the point $(0, 0)$ in the plane to the point $(8, 15)$. The terrapin travels in a straight line, while the rabbit goes in a straight line to $(8, 0)$ and then in a straight line from $(8, 0)$ to $(8, 15)$. How much farther does the rabbit travel than the terrapin?

   a. 0   b. 2   c. 4   d. 6   e. 8

4. A drawer contains 2 pairs of red gloves, 2 pairs of black gloves, and 2 pairs of yellow gloves, for a total of 12 gloves in all. Each pair of gloves consists of a left-hand glove and a right-hand glove that are not interchangeable. A number m of gloves are removed from the drawer at random. What is the least value of m that guarantees there will be a pair of gloves (that is, a right glove and a left glove) of the same color among them?

   a. 4   b. 5   c. 6   d. 7   e. 8

5. You are on an airplane traveling at 600 miles per hour. When you walk from the back of the plane to the front, it takes 1 minute and your GPS watch says you are traveling at 602 miles per hour. How long is the plane? (Note: 1 mile is 5280 feet)

   a. 100 feet   b. 144 feet   c. 156 feet   d. 176 feet   e. 280 feet

6. Four candidates ran for the office of head enumerator. Each received a two-digit prime number of votes, and no two got the same number of votes. What is the largest possible number of votes for the candidate who finished last?

   a. 83 b. 79 c. 73 d. 71 e. 67

7. A termite tells its parent that it just ate some logs. The parent asks how many. The termite replies $8(\log 20 + \log 50)$, where logarithms are to the base 10. How many logs did the termite eat?

   a. 8 b. 16 c. 24 d. 56 e. 78

8. The primes 2, 3, 5, 7, 11, 13, 17, and 19 are put in a box and Bob Almost-Square-Pants draws out two distinct primes p and q at random. Bob wins if the product pq is one less than a square. What is the probability that Bob wins?

   a. 1/7 b. 1/6 c. 1/5 d. 1/4 e. 1/3

9. Banach, Cauchy, and Dedekind choose numbers b, c, and d, and they form the polynomial $f(x) = ax^3 + bx^2 + cx + d$, where a has not yet been chosen (because Abel overslept). Tartaglia points out that if $a = 3$ then $f(3) = 0$. Turing says that if $a = 2$ then $f(2) = 0$. Omar Khayyam says that if $a = 1$, then $f(1) = 0$. Zeno declares that if $a = 0$ then $f(x) = 0$ has one solution $x > 0$. What is this positive solution?

   a. 3/4 b. 1 c. 1/2 d. 4 e. 6/5

10. In a used bookstore, you find a calendar from the previous century (that is, the 1900s) but the numbers are written in base b (where b is a positive integer). The year on the calendar is 123_b. What year is it in base 10?

    a. 1910 b. 1938 c. 1951 d. 1977 e. 1985

11. How many solutions are there to the equation $\sin(x) + \cos(x) = 0.1$ with $0 \le x < 2\pi$? (x is in radians)

    a. 0 b. 1 c. 2 d. 4 e. 6

12. Two circles with radii r and R with $r < R$ have the same center O. It is known that there is a triangle with vertices on the larger circle whose sides are tangent to the smaller circle. The ratio R/r is equal to

    a. 2 b. $\sqrt{5}$ c. $\sqrt{3}$ d. $\sqrt{2}$ e. cannot be determined from the given information

13. Suppose ABC is an isosceles right triangle with hypotenuse BC, and let D be a point on side AB such that the angles ACD and BCD are equal. The length $|BD|$ of segment BD is equal to

    a. $|BC| - |AC|$ b. $|AB| + |AC| - |BC|$ c. $\frac{2}{3}|AB|$ d. $\frac{2}{5}|BC|$ e. $\frac{|BC|}{|AB| + |BC|}$

14. Calculate

$$1^2 + 2^2 - 3^2 - 4^2 + 5^2 + 6^2 - 7^2 - 8^2 + \cdots + 2021^2 + 2022^2$$

a. -2023 b. 2023 c. 2045253 d. 2047275 e. 4090505

15. Alice needs to place 8 balls in a box - some red and some green. The balls will be mixed and then three will be pulled out (without replacement). (Each set of three is equally likely to be pulled out). Alice wants to maximize the chances that one of those three is red and the other two are green. How many red balls should she place in the box originally? Select the best answer.

a. 2 b. 3 c. 2 or 3 d. 3 or 4 e. 4

16. Consider a 4×4 square with the corners at $(0,0),(0,4),(4,4)$, and $(4,0)$. A frog starts at the center, $(2,2)$, and jumps in either vertical or horizontal direction by a distance of one, thus ending at an integer point after every jump. It stops when it reaches one of the sides of the square. (For example, the frog could jump from $(2,2)$ to $(2,3)$, then to $(3,3)$, then to $(3,2)$, then to $(2,2)$, then to $(2,1)$ and then to $(2,0)$, where it stops). In how many ways can the frog choose its trajectory if it wants to reach one of the sides after making no more than 4 jumps?

a. 36 b. 40 c. 44 d. 48 e. 52

17. How many pairs of integers a,b with $-20 \leq a,b \leq 20$ are there for which the two equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ have at least one common real root?

a. 38 b. 40 c. 80 d. 79 e. 78

18. How many integers $1 < n < 100$ are there with the property that the product of all the positive integer factors of n is equal to n^3?

a. 12 b. 13 c. 14 d. 15 e. 16

19. A triangle ABC with side lengths 13, 14, and 15 is given. A point P in the plane of $\triangle ABC$ is said to be *neutral* whenever $[PAB] = [PBC] = [PCA]$. Let $\mathcal{N}$ be the set of all neutral points and let $\mathcal{C}$ be the smallest convex polygon for which all points of $\mathcal{N}$ are either on or inside $\mathcal{C}$. What is the area of $\mathcal{C}$? (Note: $[XYZ]$ denoted the area of triangle XYZ.)

a. 336 b. 360 c. 400 d. 420 e. 440

20. In a triangle ABC we know $\angle A < \angle B < \angle C < 90^\circ$. Assume the angle between the altitude and the angle bisector at vertex A is 6°, and the angle between the altitude and the angle bisector at vertex B is 12°. Find the measure of the angle between the altitude and the angle bisector at vertex C.

a. 3 b. 4 c. 6 d. 8 e. 12

21. Consider the equation $ax^4 + bx^3 + x^2 + bx + a = 0$, where a,b are real numbers, and $a > 1/2$. What is the maximum possible value of $a+b$ for which there is at least one positive real root of

the above equation?

a. 1 b. 1/2 c. 0 d. $-1/2$ e. -1

22. Four vertices of a square are on four sides of a regular hexagon of area one, in such a way that two opposite sides of the square are parallel to two opposite sides of the hexagon. Compute the area of the square.

a. $11 - 6\sqrt{3}$ b. $\dfrac{8\sqrt{3}-12}{3}$ c. $\dfrac{\sqrt{3}}{3}$ d. $\dfrac{\sqrt{5}-1}{2}$ e. $\dfrac{2}{3}$

23. Let p be a prime factor of $64^7 + 128^3 + 1$. Which of the following can be the sum of digits of p?

a. 5 b. 7 c. 8 d. 10 e. 13

24. Suppose a, b, c, d are four real numbers satisfying all of the following,

$$\begin{cases} ad - bc = 1 \\ a^2 + b^2 + c^2 + d^2 = 2 \\ a + 2b + 3c + 4d = 5 \end{cases}$$

The smallest possible value of a can be written as a fraction $\dfrac{m}{n}$ in reduced form, where m, n are positive integers. What is $m + n$?

a. 2 b. 15 c. 25 d. 27 e. 34

25. For every positive integer n, let $\varphi(n)$ be the number of positive integers not exceeding n that are relatively prime to n, and let $\omega(n)$ be the number of prime numbers not exceeding n that do not divide n. For how many positive integers $1 < n < 1000$ do we have $\varphi(n) = \omega(n) + 1$?

a. 7 b. 8 c. 9 d. 10 e. 11