THE 43rd ANNUAL (2022) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART I SOLUTIONS

1. These 100 people have a total of 100×22 cookies. The total number of cookies for $100 + N$ people is $(100 + N) \times 20$. Therefore, we must have

$$100 \times 22 = (100 + N) \times 20 \Rightarrow 110 = 100 + N \Rightarrow N = 10.$$

The answer is **a**.

2. A point is purple if it is on the line $y = x + 6$ and on the parabola $y = x^2$. This yields the equation $x^2 = x + 6$, which can be written as $x^2 - x - 6 = 0$, and can be factored as $(x - 3)(x + 2) = 0 \Rightarrow x = 3, -2$. The answer is **c**.

3. By the distance formula, the length of the path of terrapin is $\sqrt{8^2 + 15^2} = 17$ and the length of rabbit's path is $8 + 15 = 23$. Thus, the rabbit travels $23 - 17 = 6$ units farther. The answer is **d**.

4. Since there are six pairs of gloves, it is possible to select six non-matching gloves. Once we choose seven gloves, we would have to have a pair of matching gloves. The answer is **d**.

5. Since the plane travels at 600 mph and the GPS watch shows their speed at 602 mph, their speed must be 2 mph. Therefore, the length of the plane is $\frac{2}{60} \times 5280 = 176$ feet. The answer is **d**.

6. The largest two-digit prime numbers are 97, 89, 83, 79. The answer is **b**.

7. $8(\log 20 + \log 50) = 8 \log(20 \times 50) = 8 \log(1000) = 8 \times 3 = 24$. The answer is **c**.

8. Suppose $p < q$ and $pq = a^2 - 1$ for some positive integer a. Then, $pq = (a - 1)(a + 1)$. Since $pq \geq 6$, we must have $a \geq 3$. Therefore, both $a - 1$ and $a + 1$ are more than 1, and since p, q are both prime, we must have $p = a - 1$ and $q = a + 1$. This yields $q - p = 2$. Furthermore, if $q - p = 2$, then such an integer a exists. Among the given primes, there are four pairs of primes that differ by 2. (They are $(3, 5), (5, 7), (11, 13),$ and $(17, 19)$). Since there are eight primes, we can select two of them in $\binom{8}{2} = 28$ ways. Therefore, the probability of choosing a pair p, q for which pq is one less than a perfect square is $4/28 = 1/7$. The answer is **a**.

9. The given assumption yields the following system of equations:

$$\begin{cases} 81 + 9b + 3c + d = 0 \\ 16 + 4b + 2c + d = 0 \\ 1 + b + c + d = 0 \end{cases}$$

Subtracting the last equation from the other two yields

$$\begin{cases} 80 + 8b + 2c = 0 \\ 15 + 3b + c = 0 \end{cases}$$

Subtracting twice the second equation from the first one we obtain $50 + 2b = 0$, or $b = -25$. Substituting into $15 + 3b + c = 0$ we obtain $c = 60$ and hence $d = -36$. When $a = 0$, we obtain the equation $-25x^2 + 60x - 36 = 0$. This can be factored as $-(5x-6)^2 = 0$. Thus, $x = 6/5$. The answer is **e**.

10. This year is $3+2b+b^2$. For this year to be in the 1900s we need to have $1900 \leq 3+2b+b^2 < 2000$. This can be simplified to $1898 \leq (b+1)^2 < 1998$. Therefore, $b = 43$ and the year is 1938. The answer is **b**.

11. **First solution.** The point $(\cos(x), \sin(x))$ is on the unit circle. In order for it to also satisfy $\sin(x) + \cos(x) = 0.1$, it must lie on the line $x + y = 0.1$. This line and the unit circle intersect at two points. Therefore, the answer is **c**.

    Second solution. Dividing both sides by $\sqrt{2}$ we obtain

$$\frac{\sin(x)}{\sqrt{2}} + \frac{\cos(x)}{\sqrt{2}} = \frac{0.1}{\sqrt{2}} \Rightarrow \cos\left(\frac{\pi}{4}\right)\sin(x) + \sin\left(\frac{\pi}{4}\right)\cos(x) = \frac{0.1}{\sqrt{2}} \Rightarrow \sin(x+\pi/4) = \frac{0.1}{\sqrt{2}}$$

    There are two solutions to this equation, one in the first quadrant and one in the second quadrant. The answer if **c**.

12. Call the triangle ABC, and let H be the foot of perpendicular from O to one side AB of the triangle. since the circumcenter and incenter of ABC coincide, ABC must be equilateral. By assumption $r = |OH|$ and $R = |OA|$. Since AO is the angle bisector of $\angle BAC$, we have $\angle HAO = 30^\circ$. Thus, $\dfrac{|OA|}{|OH|} = \csc(30^\circ) = 2$. The answer is **a**.

13. Setting $a = |AB| = |AC|$, since the triangle ABC is isosceles, $|BC| = a\sqrt{2}$. Using the Angle Bisector Theorem, we obtain

$$\begin{aligned} \frac{|BD|}{|AD|} = \frac{a\sqrt{2}}{a} \quad &\Rightarrow \quad \frac{|BD|}{a - |BD|} = \sqrt{2} \\ &\Rightarrow \quad |BD| = a\sqrt{2} - |BD|\sqrt{2} \\ &\Rightarrow \quad |BD| = \frac{a\sqrt{2}}{1+\sqrt{2}} = a\sqrt{2}(\sqrt{2}-1) = 2a - a\sqrt{2} \\ &\Rightarrow \quad |BD| = |AB| + |AC| - |BC|. \end{aligned}$$

    The answer is **b**.

14. Rearranging the terms and using difference of squares we obtain the following:

$$\begin{aligned} &1 + 4 + \sum_{k=1}^{505}(-(4k-1)^2 + (4k+1)^2) + \sum_{k=1}^{505}(-(4k)^2 + (4k+2)^2) \\ &= 5 + \sum_{k=1}^{505} 2(8k) + \sum_{k=1}^{505} 2(8k+2) = 5 + (\sum_{k=1}^{505} 32k) + 4 \times 505 \\ &= 2025 + 32\frac{505 \times 506}{2} = 4090505 \end{aligned}$$

    The answer is **e**.

15. Suppose there are r red balls and $8-r$ green balls. The chance of selecting 1 red and 2 green balls is,

$$\frac{r\binom{8-r}{2}}{\binom{8}{3}} = \frac{r(8-r)(7-r)}{112}$$

Therefore, in order to maximize this, we need to maximize $r(8-r)(7-r)$. After evaluating this expression for $r = 1, 2, 3, 4, 5, 6$ we see that $r = 2$ and $r = 3$ both yield the maximum value of 60. The answer is **c**.

16. There are four possibilities for the first move. By symmetry we will find the number of frog's trajectories that start with the move $(2, 2) \to (3, 2)$ and multiply the answer by 4. We will count these based on the number of moves after the first move. For that we will use R, L, U, D for right, left, up, and down, respectively.

One move: The only possibility is "R". So there is only 1 possible move.
Two moves: The possibilities are UR, UU, DR, DD. So, there are four possible moves.
Three moves: The possibilities are ULU, UDR, DLD, DUR, LLLL, LRR, LUU, LDD. So, there are eight possibilities.

Therefore, the number of possibilities is $4 \times (1 + 4 + 8) = 52$. The answer is **e**.

17. Subtracting the equations we obtain $ax + b - bx - a = 0$, which yields $(x-1)(a-b) = 0$, and hence $x = 1$ or $a = b$. If $x = 1$ is a common root of the equations, we must have $1 + a + b = 0$, or $a = -b - 1$. We could have $-20 \leq b \leq 19$. So we obtain 40 different pairs of (a, b).

When $a = b$, we need the discriminant of $x^2 + ax + a = 0$ to be nonnegative. Which implies $a^2 - 4a \geq 0$. Therefore, $a \leq 0$ or $a \geq 4$. This yields $21 + 17 = 38$ pairs of (a, b). Adding these up, we conclude there are 78 pairs of (a, b). The answer is **e**.

18. For simplicity, let $d(k)$ be the number of positive divisors of k. Let p be a prime factor of n and write $n = p^\alpha m$, where m is an integer, relatively prime to p. Each factor p^β with $0 \leq \beta \leq \alpha$ appears in exactly $d(m)$ positive divisors of n. Therefore, the exponent of p in the product of the positive divisors of p is,

$$\sum_{\beta=0}^{\alpha} \beta d(m) = \frac{\alpha(\alpha+1)d(m)}{2}.$$

By assumption, we would need this to be 3α. This implies $(\alpha+1)d(m) = 6$. This yields three possibilities for n.

Case I. $n = p^5$. In that case, $n = 32$ is the only integer in the given range.

Case II. $n = pq^2$ for distinct primes p, q. This yields the following answers,

$$n = 2\cdot 3^2, 2\cdot 5^2, 2\cdot 7^2, 3\cdot 2^2, 3\cdot 5^2, 5\cdot 2^2, 5\cdot 3^2, 7\cdot 2^2, 7\cdot 3^2, 11\cdot 2^2, 11\cdot 3^2, 13\cdot 2^2, 17\cdot 2^2, 19\cdot 2^2, 23\cdot 2^2.$$

There are 16 possibilities. So, the answer is **e**.

19. In order for a point P inside the triangle to have the given property, the distance from P to AB must be $1/3$ the distance from C to AB. Similarly for sides AC and AB. Therefore, there is a unique point inside the triangle ABC with the given properties. If a point P outside triangle ABC and inside the $\angle BAC$ has this property, then $[ABC] = [PAB]+[PAC]-[PBC]$ Therefore, $[PAB] = [PBC] = [PAC] = [ABC]$. This implies, point P lies on the line parallel to AB and through C. Similarly, P lies on the line passing through B and parallel to AC. This yields a unique point P. Three other points can similarly be found. The area of the convex region containing all these points is four times $[ABC]$. By Heron's formula

$$[ABC] = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 7 \cdot 3 \cdot 4 = 84.$$

Therefore, the total area is 336. The answer is **a**.

20. The angle between the angle bisector of A and AB is $\frac{A}{2}$. The angle between the altitude of A and AB is $90 - B$. Thus, the angle between the angle bisector of A and the altitude at A is

$$\left|\frac{A}{2} - 90 + B\right| = \left|\frac{A + 2B - 180}{2}\right| = \left|\frac{B - C}{2}\right| = \frac{C - B}{2}$$

By assumption we have $C - B = 12$ and $C - A = 24$. Subtracting we obtain $B - A = 12$. Therefore, the angle between the angle bisector and the altitude at C is $6°$. The answer is **c**.

21. Dividing both sides by x^2 we obtain

$$ax^2 + bx + 1 + \frac{b}{x} + \frac{a}{x^2} = 0.$$

Setting $S = x + 1/x$ we obtain $S^2 = x^2 + 1/x^2 + 2$. This yields,

$$a(S^2 - 2) + bS + 1 = 0 \Rightarrow aS^2 + bS - 2a + 1 = 0.$$

Since x is positive $S \geq 2$. Therefore, the equation $aS^2 + bS - 2a + 1 = 0$ must have a root not less than 2. Note that for $S = 0$, the quadratic $aS^2 + bS - 2a + 1$ is $-2a + 1$ which is negative. Therefore, this quadratic must be nonpositive at $S = 2$. This implies $4a+2b-2a+1 \leq 0$, which shows $a + b \leq -1/2$. When $x = 1$, we obtain $a + b = -1/2$. The answer is **d**.

22. Call the hexagon $ABCDEF$. Let M, N, P, and Q be vertices of the square on sides AB, BC, DE, and EF, respectively. Assume s is the side length of the hexagon and x be the distance $|AM|$. Dropping a perpendicular from A to MQ we obtain a $30 - 60 - 90$ triangle. Therefore, $|MQ| = s+x$. Similarly, dropping a perpendicular from B to MN we obtain $|MN| = \sqrt{3}(s-x)$. Setting these equal we obtain $x = s(2-\sqrt{3})$. Therefore, the area of the square is $(s+x)^2 = s^2(12-6\sqrt{3})$. On the other hand the hexagon can be divided into six equilateral triangles with side length s. Therefore, the area of the hexagon is $1 = 6\frac{s^2\sqrt{3}}{4} = \frac{3s^2\sqrt{3}}{2}$. Since the area of the hexagon is 1, we conclude $s^2 = \frac{2}{3\sqrt{3}}$. This gives us $s^2(12-6\sqrt{3}) = \frac{24 - 12\sqrt{3}}{3\sqrt{3}} = \frac{8\sqrt{3} - 12}{3}$. The answer is **b**.

23. This integer can be written as:

$$64^7 + 128^3 + 1 = 2^{42} + 2^{21} + 1 = \frac{2^{63} - 1}{2^{21} - 1}$$

The numerator is divisible by $2^9 - 1 = 7 \cdot 73$, while taking the denominator mod $2^9 - 1$ we obtain

$$2^{21} - 1 = (2^9)^2 \cdot 2^3 - 1 \equiv 8 - 1 = 7 (\text{mod } 2^9 - 1) \Rightarrow 2^{21} - 1 \equiv 7 (\text{mod } 73).$$

Therefore, the numerator is divisible by 73, while the denominator is not. Thus, 73 divides this number. Its sum of digits is 10. The answer is **d**.

24. Multiplying the first equation by 2 and subtracting from the second we obtain $(a-d)^2+(b+c)^2 = 0$. Since a, b, c, d are all real, we obtain $a = d$ and $b = -c$. Substituting into the first and last equation, we obtain $d^2 + c^2 = 1$ and $5d + c = 5$. Solving this we obtain $c = 5 - 5d$. This implies $25 + 25d^2 - 50d + d^2 = 1$. Solving for d we obtain $a = d = 1, 12/13$. The sum of numerator and denominator of $12/13$ is 25. The answer is **c**.

25. The equality $\varphi(n) = \omega(n) + 1$ holds if and only if no composite number less than n is relatively prime to n. The smallest composite number relatively prime to n is p^2, where p is the smallest prime not dividing n. Therefore, $n < p^2$.

    If $p = 2$, then $1 < n < 4$. Therefore, $n = 2, 3$.

    If $p = 3$, then $n < 9$ and n is divisible by 2. This gives us $n = 4, 8$.

    If $p = 5$, then $n < 25$ and n must be divisible by 6. This gives us $n = 6, 12, 18, 24$.

    If $p = 7$, then $n < 49$ and n must be divisible by $2 \cdot 3 \cdot 5 = 30$. This yields $n = 30$.

    If $p = 11$, then $n < 121$ and n must be divisible by $2 \cdot 3 \cdot 5 \cdot 7 = 210$. Thus, no such solutions exist.

    If $p = 13$, then $n < 169$ and n must be divisible by $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310$. No such integer exists.

    To summarize there are nine such integers: 2, 3, 4, 6, 8, 12, 18, 24, 30. The answer is **c**.