43rd ANNUAL UNIVERSITY OF MARYLAND
HIGH SCHOOL MATHEMATICS COMPETITION
PART II
November 30, 2022, 1:00–3:00 pm

Instructions: Solutions to different problems should go on separate pages. Write down your name and the problem number on the upper left corner of all pages that you submit for grading. Show your work and justify all of your steps. Submit only what you want to be graded; no blank paper or scratch paper.

NO CALCULATORS
2 hours

1. Find a real number x for which $x\lfloor x\rfloor = 1234$.

   Note: $\lfloor x\rfloor$ is the largest integer less than or equal to x.

2. Let C_1 be a circle of radius 1, and C_2 be a circle that lies completely inside or on the boundary of C_1. Suppose P is a point that lies inside or on C_2. Suppose O_1, and O_2 are the centers of C_1, and C_2, respectively. What is the maximum possible area of ΔO_1O_2P? Prove your answer.

3. The numbers $1, 2, \ldots, 99$ are written on a blackboard. We are allowed to erase any two distinct (but perhaps equal) numbers and replace them by their nonnegative difference. This operation is performed until a single number k remains on the blackboard. What are all the possible values of k? Prove your answer.

   Note: As an example if we start from $1, 2, 3, 4$ on the board, we can proceed by erasing 1 and 2 and replacing them by 1. At that point we are left with $1, 3, 4$. We may then erase 3 and 4 and replace them by 1. The last step would be to erase $1, 1$ and end up with a single 0 on the board.

4. Let a, b be two real numbers so that $a^3 - 6a^2 + 13a = 1$ and $b^3 - 6b^2 + 13b = 19$. Find $a + b$. Prove your answer.

5. Let m, n, k be three positive integers with $n \geq k$. Suppose $A = \prod_{1\leq i\leq j\leq m} \gcd(n+i, k+j)$ is the product of $\gcd(n+i, k+j)$, where i, j range over all integers satisfying $1 \leq i \leq j \leq m$. Prove that the following fraction is an integer

$$\frac{A}{(k+1)\cdots(k+m)}\binom{n}{k}.$$

   Note: $\gcd(a, b)$ is the greatest common divisor of a and b, and $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.