THE 43$^{\text{rd}}$ ANNUAL (2022) UNIVERSITY OF MARYLAND
HIGH SCHOOL MATHEMATICS COMPETITION
PART II SOLUTIONS

1. Find a real number x for which $x\lfloor x\rfloor = 1234$.

   Note: $\lfloor x\rfloor$ is the largest integer less than or equal to x.

   Solution. We know $\lfloor x\rfloor \le x < \lfloor x\rfloor + 1$. Assuming x is positive we obtain the inequalities, $\lfloor x\rfloor^2 \le x\lfloor x\rfloor < (\lfloor x\rfloor + 1)\lfloor x\rfloor$. Since $\lfloor x\rfloor$ is an integer, we can see $\lfloor x\rfloor = 35$. Thus, $x = \dfrac{1234}{35}$ is a solution. □

2. Let C_1 be a circle of radius 1, and C_2 be a circle that lies completely inside or on the boundary of C_1. Suppose P is a point that lies inside or on C_2. Suppose O_1, and O_2 are the centers of C_1, and C_2, respectively. What is the maximum possible area of ΔO_1O_2P? Prove your answer.

   Solution. First, note that the area of a triangle ABC is $\frac{1}{2}|BC||AH|$, where AH is the height from A to side BC. Since $|AH| \le |AB|$, we know $[ABC] \le \frac{1}{2}|BC||AB|$.

   Let r be the radius of C_2. Since C_2 is inside C_1, $|O_1O_2| \le 1 - r$. By what we showed above, the area of O_1O_2P is less than or equal to

   $$\frac{1}{2}|O_1O_2||O_2P| \le \frac{1}{2}(1-r)r = \frac{1}{2}[1/4 - (r-1/2)^2] \le 1/8.$$

   Note also that if C_2 has radius $1/2$ and is tangent to C_1 and the angle $\angle O_1O_2P = 90^\circ$, the area of O_1O_2P is equal to $1/8$. Therefore, the answer is $1/8$. □

3. The numbers $1, 2, \ldots, 99$ are written on a blackboard. We are allowed to erase any two distinct (but perhaps equal) numbers and replace them by their nonnegative difference. This operation is performed until a single number k remains on the blackboard. What are all the possible values of k? Prove your answer.

   As an example if we start from $1, 2, 3, 4$ on the board, we can proceed by erasing 1 and 2 and replacing them by 1. At that point we are left with $1, 3, 4$. We may then erase 3 and 4 and replace them by 1. The last step would be to erase $1, 1$ and end up with a single 0 on the board.

   Solution. We will show all possible values of k are $0, 2, 4, 6, \ldots, 98$. First, note that in the beginning there are 50 odd numbers on the board. In each step we either replace two odd numbers by an even number, two even numbers by an even numbers or one odd and one even numbers are replaced by an odd number. In other words, in each step we either reduce the number of odd numbers by 2 or we do not change the number of odd integers on the board. Since there are an even number of odd numbers to begin with, the number of odd numbers on the board will always remain even. Thus, the last number must be even. So, k must be one of $0, 2, 4, 6, \ldots, 98$.

Now, let 2ℓ be one of these even integers, i.e. $0 \leq \ell \leq 49$. Replacing each pair

$$(1,2), \ldots, (2\ell-3, 2\ell-2), (2\ell-1, 2\ell+1), (2\ell+2, 2\ell+3), \ldots, (98, 99)$$

by their difference, we obtain the following list of integers:

$$\underbrace{1, \ldots, 1}_{48 \text{ times}}, 2, 2\ell.$$

Replacing $1, 2$ with 1 we obtain the following list

$$\underbrace{1, \ldots, 1}_{48 \text{ times}}, 2\ell.$$

We may replace $1, 1$ by 0 and then $1, 0$ by 1 and then again $1, 1$ by 0. Eventually it gives us $0, 2\ell$, which can be replaced by 2ℓ. ☐

4. Let a, b be two real numbers so that $a^3 - 6a^2 + 13a = 1$ and $b^3 - 6b^2 + 13b = 19$. Find $a + b$. Prove your answer.

Solution. We note that $(a-2)^3 = a^3 - 6a^2 + 12a - 8 = 1 - a - 8$. Thus, $(a-2)^3 + (a-2) = -9$. Similarly, $(2-b)^3 + (2-b) = -9$. If $a - 2$ were more than $2 - b$, then $(a-2)^3 + (a-2)$ would be more than $(2-b)^3 + (2-b)$, which is not the case. Similarly $a - 2$ cannot be less than $2 - b$. Thus, $a - 2 = 2 - b$, which implies $a + b = 4$. ☐

5. Let m, n, k be three positive integers with $n \geq k$. Suppose $A = \prod_{1 \leq i \leq j \leq m} \gcd(n+i, k+j)$ is the product of $\gcd(n+i, k+j)$, where i, j range over all integers satisfying $1 \leq i \leq j \leq m$. Prove that the following fraction is an integer

$$\frac{A}{(k+1)\cdots(k+m)} \binom{n}{k}.$$

Solution. First, we will show that if the product of n positive integers $a_1, \ldots, a_n$ is divisible by an integer b, then the product of $\gcd(a_1, b), \ldots, \gcd(a_n, b)$ is also divisible by b. Let p be a prime factor of b and let $\alpha_1, \ldots, \alpha_n$, β be the exponents of p in the prime factorizations of $a_1, \ldots, a_n$, and b. If $\alpha_i \geq \beta$ for some i, then the exponent of p in $\gcd(a_i, b)$ and b are the same. Otherwise, the exponent of p in a_i is the same as the exponent of p in $\gcd(a_i, b)$. Thus, in both cases, the exponent of p in prime factorization of b does not exceed the exponent of p in the product $\gcd(a_1, b) \cdots \gcd(a_n, b)$.

Let $A_m = \prod_{1\le i\le j\le m} \gcd(n+i, k+j)$. We need to prove

$$\frac{A_m}{(k+1)\cdots(k+m)}\binom{n}{k}$$

is an integer. We will do so by induction on m.

For $m = 1$, $A_1 = \gcd(n+1, k+1)$. We see that

$$(n+1)\binom{n}{k} = (n+1)\frac{n!}{k!(n-k)!} = \frac{(n+1)!}{(k+1)!(n-k)!}(k+1) = (k+1)\binom{n+1}{k+1}.$$

Therefore, $(n+1)\binom{n}{k}$ is divisible by $k+1$. Since $k+1$ also divides $(k+1)\binom{n}{k}$, it must also divide the greatest common divisor of $(n+1)\binom{n}{k}$ and $(k+1)\binom{n}{k}$, which is $A_1\binom{n}{k}$. Thus, $\frac{A_1}{k+1}\binom{n}{k}$ is an integer.

Now, assume we know

$$\frac{A_m}{(k+1)\cdots(k+m)}\binom{n}{k} \text{ is an integer.}$$

$$\frac{A_m(n+1)\cdots(n+m+1)}{(k+1)\cdots(k+m)}\binom{n}{k} = \frac{A_m(n+m+1)!}{(k+m)!(n-k)!} = A_m(k+m+1)\binom{n+m+1}{k+m+1}.$$

Therefore, $\frac{A_m(n+1)\cdots(n+m+1)}{(k+1)\cdots(k+m)}\binom{n}{k}$ is a multiple of $k+m+1$. By what we showed above, the following is divisible by $k+m+1$.

$$\frac{A_m \gcd(n+1, k+m+1)\cdots\gcd(n+m+1, k+m+1)}{(k+1)\cdots(k+m)}\binom{n}{k}.$$

Dividing this by $k+m+1$, we conclude that

$$\frac{A_{m+1}}{(k+1)\cdots(k+m+1)}\binom{n}{k}$$

is an integer.

□