THE 44^{th} ANNUAL (2023) UNIVERSITY OF MARYLAND

HIGH SCHOOL MATHEMATICS COMPETITION

PART I   MULTIPLE CHOICE

For each of the following 25 questions, carefully blacken the appropriate box on the answer sheet with a #2 pencil. Do not fold, bend or write stray marks on either side of the answer sheet. Each correct answer is worth 6 points. Two points are awarded for answers left blank. Zero points are given for incorrect answers. Note that wild guessing is apt to lower your score. When the exam is over, give your answer sheet to the proctor. You may keep your copy of the questions.

NO CALCULATORS

75 MINUTES

1. An ant walks a distance $A = 10^9$ millimeters. A bear walks $B = 10^6$ feet. A chicken walks $C = 10^8$ inches. What is the correct ordering of A, B, C? (Note: There are 25.4 millimeters in an inch, and there are 12 inches in a foot.)

   a. $A < B < C$ b. $A < C < B$ c. $C < B < A$ d. $B < A < C$ e. $B < C < A$

2. Peter Rabbit is hopping along the number line, always jumping in the positive x direction. For his first jump, he starts at 0 and jumps 1 unit to get to the number 1. For his second jump, he jumps 4 units to get to the number 5. He continues jumping by jumping 1 unit whenever he is on a multiple of 3 and by jumping 4 units whenever he is on a number that is not a multiple of 3. What number does he land on at the end of his 100th jump?

   a. 297   b. 298   c. 301   d. 302   e. 303

3. Adam is walking in the city. In order to get around a large building, he walks 12 miles east and then 5 miles north, then stops. His friend Neutrino, who can go through buildings, starts in the same place as Adam but walks in a straight line to where Adam stops. How much farther than Neutrino does Adam walk?

   a. 1 mile   b. 2 miles   c. 3 miles   d. 4 miles   e. 5 miles

4. Euler is selling Mathematician cards to Gauss. Three Fermat cards plus 5 Newton cards costs 95 Euros, while 5 Fermat cards plus 2 Newton cards also costs 95 Euros. How many Euros does one Fermat card cost?

   a. 10   b. 15   c. 20   d. 30   e. 35

5. You shoot an arrow in the air. It falls to earth, you know not where. But you do know that the arrow's height in feet after t seconds is $-16t^2 + 80t + 96$. After how many seconds does the arrow hit the ground? (the ground has height 0)

   a. 2   b. 3   c. 4   d. 5   e. 6

6. Let

$$A = \log(1) + \log(2) + \log(3) + \cdots + \log(2023)$$

and let

$$B = \log(1/1) + \log(1/2) + \log(1/3) + \cdots + \log(1/2023).$$

(logs are logs base 10)

What is the value of $A + B$?

a. 0 b. 1 c. $-\log(2023!)$ d. $\log(2023!)$ e. -2023

7. Suppose $S = \{1, 2, 3, x\}$ is a set with four distinct real numbers for which the difference between the largest and smallest values of S is equal to the sum of elements of S. What is the value of x?

a. -1 b. -3/2 c. -2 d. -2/3 e. -3

8. How many positive integers less than 1 million have exactly 5 positive divisors?

a. 1 b. 5 c. 11 d. 23 e. 24

9. The Amazing Prime company ships its products in boxes whose length, width, and height (in inches) are prime numbers. If the volume of one of their boxes is 105 cubic inches, what is its surface area (that is, the sum of the areas of the 6 sides of the box) in square inches?

a. 21 b. 71 c. 77 d. 105 e. 142

10. There are 100 people in a room. Some are *wise* and some are *optimists*.

- A *wise* person can look at someone and know if they are wise or if they are an optimist.
- An *optimist* thinks everyone is wise (including themselves).

Everyone in the room writes down what they think is the number of wise people in the room. What is the smallest possible value for the average?

a. 10 b. 25 c. 50 d. 75 e. 100

11. Let S_1 be a square with side s and C_1 be the circle inscribed in it. Let C_2 be a circle with radius r and S_2 be a square inscribed in it. We are told that the area of $S_1 - C_1$ is the same as the area of $C_2 - S_2$. Which of the following numbers is closest to s/r?

a. 1 b. 2 c. 3 d. 4 e. 5

12. Suppose for real numbers a, b, c we know $a + \frac{1}{b} = 3$, and $b + \frac{3}{c} = \frac{1}{3}$. What is the value of $c + \frac{27}{a}$?

a. 1 b. 3 c. 8 d. 9 e. 21

13. The orthocenter of triangle ABC lies on its circumcircle. One of the angles of ABC must equal: (Note: The orthocenter of a triangle is the point where all three altitudes intersect.)

    a. $30°$ b. $60°$ c. $90°$ d. $120°$ e. It cannot be deduced from the given information.

14. Let $m \neq -1$ be a real number. Consider the quadratic equation

$$(m+1)x^2 + 4mx + m - 3 = 0.$$

    Which of the following <u>must</u> be true?

    (I) Both roots of this equation must be real.
    (II) If both roots are real, then one of the roots must be less than -1.
    (III) If both roots are real, then one of the roots must be larger than 1.

    a. Only (I) b. (I) and (II) c. Only (III) d. Both (I) and (III) e. (I), (II), and (III)

15. What is the least positive integer m such that the following is true?

    Given m integers between 1 and 2023, inclusive, there must exist two of them a, b such that $1 < \frac{a}{b} \leq 2$.

    a. 10 b. 11 c. 12 d. 13 e. 1415

16. How many integers between 123 and 789 have at least two identical digits, when written in base 10?

    a. 180 b. 184 c. 186 d. 189 e. 191

17. The lengths of the sides of triangle $A'B'C'$ are equal to the lengths of the three medians of triangle ABC. Then the ratio $\text{Area}(A'B'C')/\text{Area}(ABC)$ equals

    a. $\frac{1}{2}$ b. $\frac{2}{3}$ c. $\frac{3}{4}$ d. $\frac{5}{6}$ e. Cannot be determined from the information given.

18. How many ordered triples of integers (a, b, c) satisfy the following system?

$$\begin{cases} ab + c = 17 \\ a + bc = 19 \end{cases}$$

    a. 2 b. 3 c. 4 d. 5 e. 6

19. Three positive real numbers a, b, c satisfy $a^b = 343, b^c = 10, a^c = 7$. Find b^b.

    a. 1000 b. 900 c. 1200 d. 4000 e. 100

20. A strip is defined as the region between two parallel lines; the width of the strip is the distance between the two lines. Two strips of width 1 intersect in a parallelogram whose area is 2. What is the angle between the strips?

a. 15° b. 30° c. 45° d. 60° e. 90°

21. Let a, b, c, d, e be real numbers such that $a < b < c < d < e$. The least possible value of the function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = |x-a| + |x-b| + |x-c| + |x-d| + |x-e|$ is

a. $e+d+c+b+a$ b. $e+d+c-b-a$ c. $e+d+|c|-b-a$ d. $e+d+b-a$ e. $e+d-b-a$

22. A sequence $a_1, a_2, \ldots$ satisfies $a_1 = \frac{5}{2}$ and $a_{n+1} = a_n^2 - 2$ for all $n \geq 1$. Let M be the integer which is closest to a_{2023}. The last digit of M equals

a. 0 b. 2 c. 4 d. 6 e. 8

23. Assume a triangle ABC satisfies $|AB| = 1, |AC| = 2$ and $\angle ABC = \angle ACB + 90^\circ$. What is the area of ABC?

a. 6/7 b. 5/7 c. 1/2 d. 4/5 e. 3/5

24. Bob is practicing addition in base 2. Each time he adds two numbers in base 2, he counts the number of carries. For example, when summing the numbers 1001 and 1011 in base 2,

$$\begin{array}{ccccc} {\scriptstyle 1} & & {\scriptstyle 1} & {\scriptstyle 1} & \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 \end{array}$$

there are three carries (shown on the top row). Suppose that Bob starts with the number 0, and adds 111 (i.e. 7 in base 2) to it one hundred times to obtain the number 1010111100 (i.e. 700 in base 2). How many carries occur (in total) in these one hundred calculations?

a. 280 b. 289 c. 291 d. 294 e. 297

25. Suppose that S is a set of real numbers between 2 and 8 inclusive, and that for any two elements $y > x$ in S,

$$98y - 102x - xy \ \geq \ 4.$$

What is the maximum possible size for the set S?

a. 12 b. 14 c. 16 d. 18 e. 20