THE 44th ANNUAL (2023) UNIVERSITY OF MARYLAND
HIGH SCHOOL MATHEMATICS COMPETITION
PART II SOLUTIONS

Video Solutions can be found on YouTube:

https://www.youtube.com/@DrEbrahimian

1. An Indian raga has two kinds of notes: a short note, which lasts for 1 beat, and a long note, which lasts for 2 beats. For example, there are 3 ragas which are 3 beats long: 3 short notes, a short note followed by a long note, and a long note followed by a short note. How many Indian ragas are 11 beats long? Justify your answer.

Solution. (Video Solution) Let R_n be the number of Indian ragas of that are n beats long. Since the last note can be either a short or long note, we have $R_n = R_{n-1} + R_{n-2}$. We also note that $R_1 = 1$ and $R_2 = 2$. This allows us to find R_n recursively:

n	1	2	3	4	5	6	7	8	9	10	11
R_n	1	2	3	5	8	13	21	34	55	89	144

The answer is $\boxed{144}$. $\square$

2. Let $n \geq 2$ be an integer. There are n houses in a town. The distances between every pair of house is different. Every house sends a visitor to the house closest to it. Find all possible values of n (with full justification) for which we can design a town with n houses where every house is visited.

Solution. (Video Solution) We will show all possible values are all even positive integers n.

First, we will show by induction on n that if such a setting exists then n is even.

$n = 2$ is even. Given $n = 3$ houses, the houses that are closest to each other trade visitors, so the third house cannot have a visitor. This settles the basis step.

Suppose n houses are given with different distances and after each house sends a visitor to its closest house, every house gets a visitor. Assume H_1 and H_2 are the closest houses in this town. By assumption H_1 and H_2 must trade visitors. Since the remaining $n - 2$ houses are visited, by inductive hypothesis $n - 2$ must be even. Therefore, n must be even.

Now, assume n is even. Pair up houses of this town into $n/2$ pairs of twin houses. Make sure the distances between the twins are "small" and distinct, e.g. $1, 2, \ldots, n/2$, while the houses are constructed in such a way that the distance between every two house is distinct and that the closest house to every house is its twin house. This guarantees that every house is visited. $\square$

3. Let p be a prime, and $n > p$ be an integer. Prove that $\binom{n+p-1}{p} - \binom{n}{p}$ is divisible by n.

Note: $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

Solution. (Video Solution) We have the following:

$$\begin{aligned}
\tbinom{n+p-1}{p} - \tbinom{n}{p} &= \frac{(n+p-1)(n+p-2)\cdots n}{p!} - \frac{n(n-1)\cdots(n-p+1)}{p!} \\
&= \frac{n}{p}\left(\frac{(n+p-1)\cdots(n+1)}{(p-1)!} - \frac{(n-1)\cdots(n-p+1)}{(p-1)!}\right) \\
&= \frac{n}{p}\left(\binom{n+p-1}{p-1} - \binom{n-1}{p-1}\right)
\end{aligned} .$$

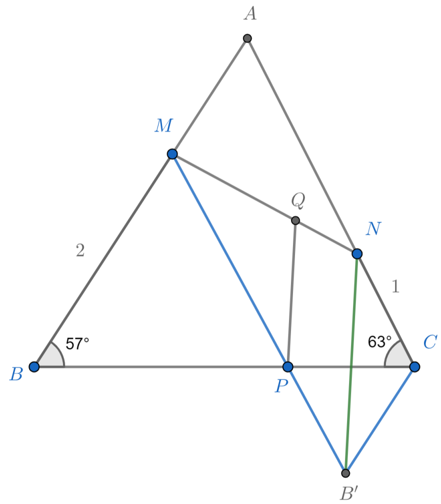
Therefore, if we show the integer $\binom{n+p-1}{p-1} - \binom{n-1}{p-1}$ is divisible by p, we are done.

$$\begin{aligned}
(p-1)!\left(\tbinom{n+p-1}{p-1} - \tbinom{n-1}{p-1}\right) &= (n+p-1)(n+p-2)\cdots(n+1) - (n-1)\cdots(n-p+1) \\
&\equiv (n-1)(n-2)\cdots(n-p+1) - (n-1)\cdots(n-p+1) = 0 \mod p
\end{aligned}$$

Since p is prime and $(p-1)!$ is not divisible by p, the integer $\binom{n+p-1}{p-1} - \binom{n-1}{p-1}$ must be divisible by p, as desired. □

4. Assume every side length of a triangle ABC is more than 2 and two of its angles are given by $\angle ABC = 57°$ and $\angle ACB = 63°$. Point P is chosen on side BC with $|BP| : |PC| = 2 : 1$. Points M, N are chosen on sides AB and AC, respectively so that $|BM| = 2$ and $|CN| = 1$. Let Q be the point on segment MN for which $|MQ| : |QN| = 2 : 1$. Find the value of $|PQ|$. Your answer must be in simplest form. Fully justify your answer.

Solution 1. (Video Solution) As shown in the diagram below, let B' be the point of intersection of line MP and the line through C parallel to AB.



Since $\triangle BMP$ and $CB'P$ are similar, we have

$$\frac{|BM|}{|CB'|} = \frac{|MP|}{|PB'|} = \frac{|BP|}{|PC|} = \frac{2}{1}.$$

Since $|BM| = 2$, we obtain $|CB'| = 1$. On the other hand, by alternate angle theorem, $\angle PCB' = \angle ABC = 57°$. Therefore, $\angle NCB' = 120°$. Using Law of Cosines in $\triangle CNB'$ we see

$$|B'N|^2 = 1^2 + 1^2 - 2 \times 1 \times 1 \cos(120°) = 3.$$

On the other hand, since $|MQ| : |QN| = |MP| : |PB'| = 2 : 1$, triangles NMB' and QMP are similar. Therefore, $|PQ| = \frac{2}{3}|B'N| = \frac{2\sqrt{3}}{3}$. □

Solution 2. We will use vectors: $\overrightarrow{AP} = \frac{2\overrightarrow{AC} + \overrightarrow{AB}}{3}$ and $\overrightarrow{AQ} = \frac{2\overrightarrow{AN} + \overrightarrow{AM}}{3}$. Subtracting we obtain:

$$\overrightarrow{PQ} = \overrightarrow{AQ} - \overrightarrow{AP} = \frac{2(\overrightarrow{AN} - \overrightarrow{AC}) + \overrightarrow{AM} - \overrightarrow{AB}}{3} = \frac{2\overrightarrow{CN} + \overrightarrow{BM}}{3}.$$

We will now use dot products to find the length of segment PQ. For that we need to find the angle between $\overrightarrow{CN}$ and $\overrightarrow{BM}$. This angle is the same as $\angle BAC$ which is $180 - 57 - 63 = 60$ degrees. Therefore,

$$|PQ|^2 = \overrightarrow{PQ} \cdot \overrightarrow{PQ} = \frac{4|CN|^2 + |BM|^2 + 4\overrightarrow{CN} \cdot \overrightarrow{BM}}{9} = \frac{4 + 4 + 4|CN|\ |BM| \cos(60°)}{9} = \frac{12}{9}.$$

This implies, $|PQ| = \frac{2\sqrt{3}}{3}$. □

Solution 3. Place the origin at point P and the x axis on the side BC. By assumption we see the coordinates of B and C are $(2x, 0)$ and $(-x, 0)$, respectively. Since $|BM| = 2$ and $|CN| = 1$, we have:

$$M = (2x - 2\cos(57^\circ), 2\sin(57^\circ)), \text{ and } N = (-x + \cos(63^\circ), \sin(63^\circ)).$$

Since $|MQ| : |QN| = 2 : 1$, we have

$$\begin{aligned} Q &= \frac{2N + M}{3} = \frac{(-2x + 2\cos(63^\circ) + 2x - 2\cos(57^\circ), 2\sin(57^\circ) + 2\sin(63^\circ))}{3} \\ &= \frac{2}{3}(\cos(63^\circ) - \cos(57^\circ), \sin(57^\circ) + \sin(63^\circ)) \end{aligned}.$$

Since P is the origin, we have

$$\begin{aligned} |PQ|^2 &= \frac{4}{9}\left((\cos(63^\circ) - \cos(57^\circ))^2 + (\sin(57^\circ) + \sin(63^\circ))^2\right) \\ &= \frac{4}{9}\left(2 - 2\cos(63^\circ)\cos(57^\circ) + 2\sin(63^\circ)\sin(57^\circ)\right) \\ &= \frac{4}{9}\left(2 - 2\cos(120^\circ)\right) = \frac{4}{9}(3) \end{aligned}$$

Therefore, $|PQ| = \dfrac{2\sqrt{3}}{3}$. □

5. Let $0 \le a_1 \le a_2 \le \cdots \le a_n \le 1$ be n real numbers with $n \ge 2$. Assume $a_1 + a_2 + \cdots + a_n \ge n - 1$. Prove that

$$a_2 a_3 \cdots a_n \ge \left(1 - \frac{1}{n}\right)^{n-1}.$$

Solution. (Video Solution) Note that if we replace a_1 by a_2, the product $a_2 \cdots a_n$ does not change and the assumption of the problem is satisfied. So, we may assume $a_1 = a_2$.

Assume for some $k < n$ we have been able to assume $a_1 = \cdots = a_k$ without increasing the product $a_2 \cdots a_n$. We will now replace each of the first $k + 1$ terms of the sequence, by their average, i.e.

$$\underbrace{a_2, \ldots, a_2}_{k \text{ times}}, a_{k+1} \text{ are replaced by } \underbrace{x, \ldots, x}_{k+1 \text{ times}}, \text{where } x = \frac{ka_2 + a_{k+1}}{k+1}.$$

We will show the new product does not exceed the old product, i.e. $a_2^{k-1} a_{k+1} \ge \left(\dfrac{ka_2 + a_{k+1}}{k+1}\right)^k$. Dividing both sides by a_2^k the above inequality is equivalent to

$$\frac{a_{k+1}}{a_2} \ge \left(\frac{k + a_{k+1}/a_2}{k+1}\right)^k \qquad (*)$$

By assumption

$$n-1 \le ka_2 + a_{k+1} + \cdots + a_n \le ka_2 + n - k \Rightarrow k-1 \le ka_2 \Rightarrow \frac{k-1}{k} \le a_2.$$

Since $a_2 \le a_{k+1} \le 1$ we have $1 \le \dfrac{a_{k+1}}{a_2} \le \dfrac{k}{k-1}$. Setting $t = \sqrt[k]{a_{k+1}/a_2}$, in order to prove $(*)$ it is enough to show if $1 \le t \le \sqrt[k]{\dfrac{k}{k-1}}$, then $t \ge \dfrac{k+t^k}{k+1}$. This inequality is equivalent to

$$t \ge \frac{k+t^k}{k+1} \Leftrightarrow (k+1)t \ge k + t^k \Leftrightarrow k(t-1) \ge t^k - t = t(t-1)(t^{k-2} + \cdots + 1)$$

Since $t \ge 1$, it is enough to show $k \ge t(t^{k-2} + \cdots + 1)$. Using the fact that $t \ge 1$ we obtain:

$$t(t^{k-2} + \cdots + 1) \le (k-1)t^k \le k, \text{ since } t^k = \frac{a_{k+1}}{a_2} \le \frac{k}{k-1}.$$

Finally, if $a_1 = a_2 = \cdots = a_n$, then $na_n \ge n-1$ which implies $a_n \ge \dfrac{n-1}{n}$ and hence, $a_2 \cdots a_n = \left(\dfrac{n-1}{n}\right)^{n-1}$. This completes the proof. □